

An Application of Maximum Principle to space-like Hypersurfaces with Constant Mean Curvature in Anti-de Sitter Space

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Abstract

In this paper, we study complete hypersurfaces with constant mean curvature in anti-de Sitter space $H_1^{n+1}(-1)$. we prove that if a complete space-like hypersurface with constant mean curvature $x : \mathbf{M} \rightarrow H_1^{n+1}(-1)$ has two distinct principal curvatures λ, μ , and $\inf|\lambda - \mu| > 0$, then x is the standard embedding $H^m(-\frac{1}{r^2}) \times H^{n-m}(-\frac{1}{1-r^2})$ in anti-de Sitter space $H_1^{n+1}(-1)$.

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1 Introduction

Let $M_1^{n+1}(c)$ be an $(n+1)$ -dimensional Lorentzian space form of constant curvature c . When $c > 0$, $M_1^{n+1}(c) = S_1^{n+1}(c)$, $(n+1)$ -dimensional de Sitter space; When $c = 0$, $M_1^{n+1}(c) = L^{n+1}$, $(n+1)$ -dimensional Lorentz-Minkowski space; When $c < 0$, $M_1^{n+1}(c) = H_1^{n+1}(c)$, $(n+1)$ -dimensional anti-de Sitter space. A hypersurface M of $M_1^{n+1}(c)$ is said to be space-like if the induced metric on M from that of the ambient space is Riemannian.

The following well-known result of the Bernstein type problem for maximal space-like hypersurfaces in $M_1^{n+1}(c)$ ($c \geq 0$) was proved by Calabi [1], Cheng-Yau [2], and Choquet-Bruhat [6]

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Theorem 1.1. [1][2] Let M be an n -dimensional complete maximal space-like hypersurface in an $(n+1)$ -dimensional Lorentian space form $M_1^{n+1}(c)(c \geq 0)$, then M is totally geodesic.

As a generalization of the Bernstein type problem, Cheng-Yau [2] and T. Ishihara [3] proved that a complete maximal space-like submanifold M^n of $M_1^{n+1}(c)(c \geq 0)$ is totally geodesic. In [3] T. Ishihara also proved the following result

Theorem 1.2. [3] Let M be an n -dimensional ($n \geq 2$) complete maximal space-like hypersurface in anti-de Sitter space $H_1^{n+1}(-1)$, then the norm square of the second fundamental form of M satisfies

$$S \leq n,$$

and $S = n$ if and only if $M^n = H^m(-\frac{m}{n}) \times H^{n-m}(-\frac{n-m}{n}), (1 \leq m \leq n-1)$.

In [4], Cao-Wei gave a new characterization of hyperbolic cylinder $M^n = H^m(-\frac{m}{n}) \times H^{n-m}(-\frac{n-m}{n})$ in anti-de Sitter space $H_1^{n+1}(-1)$.

Theorem 1.3. [4] Let M be an n -dimensional ($n \geq 3$) complete maximal space-like hypersurface with two distinct principal curvature λ and μ in anti-de Sitter space $H_1^{n+1}(-1)$. If $\inf(\lambda - \mu)^2 > 0$, then $M^n = H^m(-\frac{m}{n}) \times H^{n-m}(-\frac{n-m}{n}), (1 \leq m \leq n-1)$.

In [4], Cao-Wei also held a conjecture.

Conjecture: The only complete space-like hypersurfaces in $M_1^{n+1}(c)(c < 0)$ with constant mean curvature and two distinct principal curvatures λ and μ satisfying $\inf(\lambda - \mu)^2 > 0$ are the hyperbolic cylinders.

In this paper we investigate complete space-like hypersurfaces in $M_1^{n+1}(-1)$ with constant mean curvature and two distinct principal curvatures λ and μ satisfying $\inf(\lambda - \mu)^2 > 0$, and give an affirmative answer for the conjecture, and we have the following main theorem.

Theorem 1.4. Let $x : \mathbf{M} \rightarrow H_1^{n+1}(-1)$ be an n -dimensional ($n \geq 3$) complete space-like hypersurface in anti-de Sitter space $H_1^{n+1}(-1)$ with constant mean curvature and with two distinct principal curvatures λ, μ . If $\inf|\lambda - \mu| > 0$, then x is the standard embedding $H^m(-\frac{1}{r^2}) \times H^{n-m}(-\frac{1}{1-r^2})$ in anti-de Sitter space $H_1^{n+1}(-1)$.

Much recently, however, Wu has more general results like

Theorem 1.5. [10] The only complete space-like hypersurfaces in Lorentz-Minkowski $(n+1)$ -spaces ($n \geq 3$) of nonzero constant m th mean curvature ($m \leq n-1$) with two distinct principal curvatures λ and μ satisfying $\inf(\lambda - \mu)^2 > 0$ are the hyperbolic cylinders.

We should remind readers that Wu has used Otsuki's idea while we immediately use the maximum principle. So our proof is more natural and concise. In fact, Wu's results in [10] can be concluded from our method also.

2 Preliminaries

Let $x : \mathbf{M} \rightarrow H_1^{n+1}(-1)$ be an n -dimensional ($n \geq 3$) space-like hypersurface. Let e_1, \dots, e_n be a local orthonormal basis of \mathbf{M} with respect to the induced metric, and $\omega_1, \dots, \omega_n$ their dual form. Let ξ be the local unit normal vector field such that $\langle \xi, \xi \rangle = -1$.

Denote $x_i = e_i(x)$. Then we have the structure equations

$$dx = \sum_{i=1}^n \omega_i x_i, \quad dx_i = \sum_{j=1}^n \omega_{ij} x_j + \mathfrak{h}_i \xi + \omega_i x, \quad d\xi = \sum_{i=1}^n \mathfrak{h}_i x_i. \quad (2.1)$$

Denote $\mathfrak{h}_i = \sum_{j=1}^n h_{ij} \omega_j$, from [2] we have $h_{ij} = h_{ji}$. The curvature tensor can be expressed as Gauss equation

$$R_{ijkl} = -(h_{ik}h_{jl} - h_{il}h_{jk}) - (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \quad (2.2)$$

And Codazzi equation is

$$h_{ijk} = h_{ikj}, \quad (2.3)$$

where

$$\sum_{k=1}^n h_{ijk} \omega_k = dh_{ij} + \sum_{k=1}^n h_{kj} \omega_{ki} + \sum_{k=1}^n h_{ik} \omega_{kj}. \quad (2.4)$$

The mean curvature of M is given by $H = \frac{1}{n} \sum_i h_{ii}$. If $H = 0$, then M is said to be Maximal, and $H = \text{constant}$, then M is said to be of constant mean curvature.

We can choose an appropriate orthonormal basis e_1, \dots, e_n such that

$$h_{ij} = \lambda_i \delta_{ij},$$

where λ_i are principal curvatures.

If we suppose the hypersurface x has two distinct principal curvature and has constant mean curvature H , then choose an appropriate orthonormal basis e_1, \dots, e_n such that

$$\lambda_1 = \dots = \lambda_m = \lambda, \lambda_{m+1} = \dots = \lambda_n = \mu,$$

thus we obtain

$$m\lambda + (n - m)\mu = nH = \text{constant}, \quad (2.5)$$

Example 2.1. Hyperbolic cylinder

$$M_{m,n-m} = H^m(-\frac{1}{r^2}) \times H^{n-m}(-\frac{1}{1-r^2}), (1 \leq m \leq n-1).$$

We know (see [3]) that $M_{m,n-m}$ is a complete space-like hypersurface in $H_1^{n+1}(-1)$ with constant mean curvature H and two distinct principal curvature λ and μ , where

$$\lambda_1 = \dots = \lambda_m = \frac{1}{r}, \lambda_{m+1} = \dots = \lambda_n = \frac{1}{\sqrt{1-r^2}}.$$

Thus $M_{m,n-m}$ have constant mean curvature $H = \frac{m}{nr} + \frac{n-m}{n\sqrt{1-r^2}}$.

Now we have to consider two cases.

Case 1: $2 \leq m \leq n-2$.

In this case we make use of the following convention on the ranges of indices:

$$1 \leq i, j, k \leq m; m+1 \leq \alpha, \beta, \gamma \leq n; 1 \leq A, B \leq n.$$

Proposition 2.1. Let $x : \mathbf{M} \rightarrow H_1^{n+1}(-1)$ be an n -dimensional ($n \geq 3$) complete space-like hypersurface in anti-de Sitter space $H_1^{n+1}(-1)$ with constant mean curvature and with two distinct principal curvatures. If the multiplicities of these two distinct principal curvatures are greater than one, then x is the standard embedding $H^m(-\frac{1}{r^2}) \times H^{n-m}(-\frac{1}{1-r^2})$ in anti-de Sitter space $H_1^{n+1}(-1)$.

Proof. Letting $i \neq j$ or $\alpha \neq \beta$ in equation(2.4), there is

$$\begin{aligned} \sum_A h_{ijA} \omega_A &= dh_{ij} + \sum_A h_{Aj} \omega_{Ai} + \sum_A h_{iA} \omega_{Aj} = \lambda(\omega_{ij} + \omega_{ji}) = 0, \\ \sum_A h_{\alpha\beta A} \omega_A &= dh_{\alpha\beta} + \sum_A h_{A\beta} \omega_{A\alpha} + \sum_A h_{\alpha A} \omega_{A\beta} = \lambda(\omega_{\alpha\beta} + \omega_{\beta\alpha}) = 0. \end{aligned}$$

That is, when $i \neq j$ and $\alpha \neq \beta$, we have $h_{ijA} = 0, h_{\alpha\beta A} = 0, \forall A$.

By letting $i = j$ or $\alpha = \beta$ in equation(2.4), there is

$$d\lambda = \sum_{A=1}^n h_{iiA} \omega_A = h_{iii} \omega_i + \sum_{\alpha=m+1}^n h_{ii\alpha} \omega_\alpha, \forall i, \quad (2.6)$$

$$d\mu = \sum_{A=1}^n h_{\alpha\alpha A} \omega_A = h_{\alpha\alpha\alpha} \omega_\alpha + \sum_{i=1}^n h_{i\alpha\alpha} \omega_i, \forall \alpha. \quad (2.7)$$

Since $2 \leq m \leq n-2$, equations (2.6) and (2.7) come to

$$d\lambda = \sum_{\alpha=m+1}^n h_{ii\alpha} \omega_\alpha, d\mu = \sum_{i=1}^n h_{i\alpha\alpha} \omega_i. \quad (2.8)$$

Since $m\lambda + (n-m)\mu = nH = \text{constant}$, we know that

$$md\lambda + (n-m)d\mu = 0. \quad (2.9)$$

Combining with equations (2.8) and (2.9), we get

$$\lambda = \text{constant}, \mu = \text{constant}, h_{ii\alpha} = 0, h_{i\alpha\alpha} = 0, \forall i, \alpha. \quad (2.10)$$

Then we complete the proof of Lemma 2.1.

For any i and α in equation(2.4), we have

$$\sum_{A=1}^n h_{i\alpha A} \omega_A = h_{ii\alpha} \omega_i + h_{i\alpha\alpha} \omega_\alpha = 0 = (\lambda - \mu) \omega_{i\alpha}.$$

That is,

$$\omega_{i\alpha} = 0, \text{ and } (M, I) = (M_1, I_1) \times (M_2, I_2).$$

We assert that M_1, M_2 have constant curvature. For $i \neq j$ and $\alpha \neq \beta$, from equation(2.2) the sectional curvature of (M_1, I_1) and (M_2, I_2) is

$$K(e_i \wedge e_j) = R_{ijij} = -1 - \lambda^2, K(e_\alpha \wedge e_\beta) = R_{\alpha\beta\alpha\beta} = -1 - \mu^2, \quad (2.11)$$

respectively.

On other hands from $K(e_i \wedge e_\alpha) = R_{i\alpha i\alpha} = -1 - \lambda\mu = 0$. Then we know that when $2 \leq m \leq n-2$, $x(M)$ is locally Lorentz congruent to the standard embedding $H^m(-\frac{1}{r^2}) \times H^{n-m}(-\frac{1}{1-r^2}) \subset H_1^{n+1}(-1)$.

Thus we complete the proof of proposition.

Case 2: $m = n-1$.

In this case we make use of the following convention on the ranges of indices:

$$1 \leq i, j, k \leq n-1; 1 \leq A, B \leq n.$$

From (2.5), we can suppose that

$$(n-1)\lambda + \mu = nH, \quad \lambda - \mu = n(\lambda - H) \neq 0. \quad (2.12)$$

Similarly, we have

$$h_{ijA} = 0, \forall A, i \neq j; \quad d\lambda = h_{iii}\omega_i + h_{iin}\omega_n, d\mu = \sum_{i=1}^{n-1} h_{inn}\omega_i + h_{nnn}\omega_n, \forall i. \quad (2.13)$$

Because $n-1 \geq 2$, from equations (2.13) and (2.9), we get

$$d\lambda = h_{iin}\omega_n, \forall i; \quad d\mu = h_{nnn}\omega_n, \quad h_{inn} = 0, h_{iii} = 0, \forall i, \quad (2.14)$$

Equation (2.11) comes to

$$\omega_{in} = \frac{1}{\lambda - \mu} h_{iin}\omega_i = \frac{1}{n(\lambda - H)} h_{iin}\omega_i. \quad (2.15)$$

And we assert that the integral curve of e_n is a geodesic because

$$\nabla_{e_n} e_n = \sum_{i=1}^{n-1} \omega_{ni}(e_n) e_i = - \sum_{i=1}^{n-1} \frac{1}{\lambda - \mu} h_{iin}\omega_i(e_n) e_i = 0.$$

We also have $d\omega_n = \sum_{i=1}^{n-1} \omega_{ni} \wedge \omega_i = 0$. It means that there exists an arc parametric s of the integral curve of e_n such that $\omega_n = ds$. Since M is complete, the arc s tends to infinity.

If we denote $\dot{f} = \frac{df}{ds}$ for any smooth function $f = f(s)$ on the integral curve of e_n , it follows from equation (2.14) that

$$d\lambda = \dot{\lambda}ds, \quad h_{iin} = \dot{\lambda}, \quad \forall i. \quad (2.16)$$

From equations (2.15) and (2.16) it follows that

$$\omega_{in} = \frac{\dot{\lambda}}{n(\lambda - H)} \omega_i, \quad (2.17)$$

Exploring into

$$d\omega_{in} = d\left(\frac{\dot{\lambda}}{n(\lambda - H)} \omega_i\right) = \sum_{j=1}^{n-1} \omega_{ij} \wedge \omega_{jn} - \frac{1}{2} \sum_{A,B=1}^n R_{inAB} \omega_A \wedge \omega_B \quad (2.18)$$

and collecting the items of $\omega_i \wedge \omega_n$, we get

$$\frac{\ddot{\lambda}}{n(\lambda - H)} - \frac{n+1}{n^2} \frac{(\dot{\lambda})^2}{(\lambda - H)^2} = R_{inin} = -1 - \lambda\mu. \quad (2.19)$$

We introduce the following generalized Liouville-type theorem (see Choi-Kwon-Sun [5]) in order to prove our main theorem.

Theorem 2.1. ([5]) Let M be a complete Riemannian manifolds whose Ricci curvature is bounded from below. Let F be any formula of the variable f with constant coefficients such that

$$F(f) = c_0 f^n + c_1 f^{n-1} + \cdots + c_k f^{n-k} + c_{k+1},$$

where $n > 1, 1 \geq n - k \geq 0$ and $c_0 > c_{k+1}$. If a C^2 -nonnegative function f satisfies

$$\Delta f \geq F(f),$$

then we have

$$F(f_1) \leq 0,$$

where f_1 denotes the supremum of the given function.

3 Proof of the main theorem

In order to complete the proof of our main theorem, we only consider Case 2. At first, we prove the following key lemma

Lemma 3.1. Let $x : \mathbf{M} \rightarrow H_1^{n+1}(-1)$ be an n -dimensional ($n \geq 3$) complete space-like hypersurface in anti-de Sitter space $H_1^{n+1}(-1)$ with constant mean curvature and two distinct principal curvatures. If one of two principal curvatures is simple, then Ricci curvature of M is negative semi-definite.

Proof. From Gauss equation (2.2) and $(n-1)\lambda + \mu = nH$ we get that

$$\begin{aligned} Ric_{nn} &= (n-1)[\mu^2 - nH\mu - 1] = (n-1)[(\mu - \frac{n}{2}H)^2 - (\frac{n^2}{4}H^2 + 1)], \\ Ric_{ii} &= -(n-1) - nH\lambda + \lambda^2 = (\lambda - \frac{n}{2}H)^2 - (\frac{n^2}{4}H^2 + n - 1), 1 \leq i \leq n-1. \end{aligned}$$

Thus we have

$$Ric_{nn} \geq -(n-1)(\frac{n^2}{4}H^2 + 1), Ric_{ii} \geq -(\frac{n^2}{4}H^2 + n - 1), 1 \leq i \leq n-1,$$

so Ricci curvature of M is bounded from below.

From (2.14) we have

$$\Delta(\lambda - H) = \sum_A (e_A e_A - \nabla_{e_A} e_A)(\lambda - H) = \ddot{\lambda} - \frac{(n-1)\dot{\lambda}}{n(\lambda - H)},$$

from this above formula and (2.19) and $(n-1)\lambda + \mu = nH$ we obtain

$$\begin{aligned} \Delta(\lambda - H) &= \frac{2}{n}(\dot{\lambda})^2 \\ &\quad + n(n-1)(\lambda - H)^3 + n(n-2)H(\lambda - H)^2 - n(1 + H^2)(\lambda - H). \end{aligned} \tag{3.1}$$

We define the formula of the variable x with constant coefficients

$$F(x) = n(n-1)x^3 + n(n-2)Hx^2 - n(1 + H^2)x.$$

Then $C_0 = n(n-1) > C_3 = 0$. From (3.1) we have

$$\Delta(\lambda - H) = \frac{2}{n}(\dot{\lambda})^2 + F(\lambda - H) \geq F(\lambda - H).$$

If necessary, take $\tilde{\xi} = -\xi$ as local unit normal vector field of M , we can assume that $\lambda - H > 0$. So

$$\sup(\lambda - H) > 0. \tag{3.2}$$

From generalized Liouville-type theorem [5] we have

$$F(\sup(\lambda - H)) \leq 0. \tag{3.3}$$

From Gauss equation (2.2) and $(n-1)\lambda + \mu = nH$ we get

$$F(\lambda - H) = n(\lambda - H)R_{inin}. \tag{3.4}$$

From (3.2) and (3.4) we obtain

$$(n-1)(\sup(\lambda-H))^2 + (n-2)H\sup(\lambda-H) - (1+H^2) \leq 0. \quad (3.5)$$

Let $f(x) = (n-1)x^2 + (n-2)Hx - (1+H^2)$, then $\lim_{x \rightarrow \infty} f(x) = +\infty$.

Since $\lambda - H > 0$ and (3.5) we obtain

$$f(\lambda-H) \leq 0, \text{ i.e., } R_{inin} \leq 0. \quad (3.6)$$

Thus from Gauss equation and (3.6) we get that

$$\begin{aligned} Ric_{ii} &= \sum_{j \neq i} R_{ijij} + R_{inin} = -1 - \lambda^2 + R_{inin} \leq 0, \\ Ric_{nn} &= \sum_i R_{inin} = (n-1)R_{inin} \leq 0. \end{aligned}$$

Thus we complete the proof of Lemma 3.1.

If we denote $w = |\lambda - H|^{-\frac{1}{n}}$, it follows that

$$\ddot{w} + wR_{inin} = 0. \quad (3.7)$$

While

$$R_{inin} = \frac{1}{n-1}R_{nn} = -1 - \lambda\mu, \quad (n-1)\lambda + \mu = nH, \quad \lambda = H \pm w^{-n},$$

we have

$$\lambda\mu = H^2 \pm (2-n)Hw^{-n} + (1-n)w^{-2n}, \quad (3.8)$$

$$\ddot{w} - w[1 + H^2 \pm (2-n)Hw^{-n} + (1-n)w^{-2n}] = 0. \quad (3.9)$$

The left hand side of equation (3.9) multiplied by $2\dot{w}$ is precisely the derivative of the left hand side of the following equation

$$\dot{w}^2 - w^2[1 + (H \pm w^{-n})^2] = C = \text{constant}. \quad (3.10)$$

Since $\ddot{w} = -wR_{inin} = -\frac{1}{n-1}wR_{nn}$ is positive, we know that \dot{w} is monotone. Because $\inf |\lambda - \mu| > 0$, $\sup \{w(s) | -\infty < s < +\infty\}$ is a bounded number. Then $\lim_{s \rightarrow +\infty} \dot{w}$ or $\lim_{s \rightarrow -\infty} \dot{w}$ cannot be infinity. We assert that $\lim_{s \rightarrow \infty} \dot{w} = 0$.

In fact, if we suppose that $\lim_{s \rightarrow +\infty} \dot{w} = a > 0$, then $\lim_{s \rightarrow +\infty} w = +\infty$. Therefore we immediately know that equation (3.10) cannot hold when s tends to infinity. On the other hand, if we suppose that $\lim_{s \rightarrow +\infty} \dot{w} = a < 0$, then $\lim_{s \rightarrow +\infty} w = -\infty$. But we know $w > 0$, which is a contraction. Therefore, $\lim_{s \rightarrow \infty} \dot{w} = 0$. Adding the monotonicity of \dot{w} , it follows that $\dot{w} \equiv 0$. That is, λ is constant, and so as μ . Similar to the discuss in case 1, we know that when $m = n-1$, $x(M)$ is locally Lorentz congruent to the standard embedding $H^{n-1}(-\frac{1}{r^2}) \times H^1(-\frac{1}{1-r^2}) \subset H^{n+1}$.

Thus we complete the proof of Theorem 1.4.

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